THE EFFECTS OF INTERNAL AND EXTERNAL FRICTION ON THE DYNAMIC STABILITY OF BARS

(O VLIIANII VNESHNEGO I VNUTRENNEGO TRENIIA NA DINAMICHESKUIU USTOICHIVOST' STERZHNEI)

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K. P. KOVALENKO
(Odessa)

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Many investigations of problems dealing with parametric resonance (without considering resistance forces) lead to the necessity of finding boundedness conditions for the solution of equations of the type

$$u''(\tau) + q(\tau)u(\tau) = 0 \qquad (0.1)$$

where $q(\tau)$ is a real periodic function of period $T = 2\pi/\omega$. Assuming

$$\tau = 2t/\omega, \qquad \lambda = 4/\omega^2, \qquad p(t) = q(2t/\omega), \qquad y(t) = u(2t/\omega)$$

equation (0.1) may be reduced to the following form:

$$\frac{d^2y}{dt^2} + \lambda p(t)y = 0 \qquad (0.2)$$

where $p(t)$ is a real periodic function of period π, and λ is a certain parameter, inversely proportional to the square of frequency of parametric disturbance of which $p(t)$ is independent.

The characteristic function $A(\lambda)$ of equation (0.2) is

$$A(\lambda) = \frac{1}{2}\{\varphi(\pi, \lambda) + \psi'(\pi, \lambda)\}$$

where $\phi(t, \lambda)$ are solutions of equation (0.2) satisfying initial conditions

$$\varphi(0, \lambda) = 1, \qquad \varphi'(0, \lambda) = 0, \qquad \psi(0, \lambda) = 0, \qquad \psi'(0, \lambda) = 1$$

As is known, the entire real λ-axis is decomposed into open intervals, so-called zones of stability, in which all solutions of equation (0.2) are bounded on the real t-axis, and into a supplementary set of points, which generally speaking are isolated points, and also into closed intervals (zones of instability).

Liapunov [1,2] (see also [3]), investigated the distribution of stability zones, on the λ-axis, requiring the analysis of relative distribution of the roots of the two equations

$$A(\lambda) - 1 = 0, \qquad A(\lambda) + 1 = 0 \tag{0.3}$$

Let us designate by $\lambda_n^{(1)}$ the characteristic numbers of the periodic boundary value problem

$$\text{(I)} \quad y''(t) + \lambda p(t) y(t) = 0, \quad y(0) = y(\pi), \quad y'(0) = y'(\pi)$$

and by $\lambda_n^{(2)}$ the characteristic numbers of the "semi-periodic" boundary value problem.

$$\text{(II)} \quad y''(t) + \lambda p(t) y(t) = 0, \quad y(0) = -y(\pi), \quad y'(0) = -y'(\pi)$$

Obviously, $\lambda_n^{(1)}$ is also the root of the first equation in (0.3) and $\lambda_n^{(2)}$ is the root of the second equation in (0.3).

The results obtained by Liapunov may be incorporated into the following theorem:

Liapunov's Theorem. If

$$L = \int_0^\pi p(t)\,dt \neq 0$$

the characteristic numbers of the boundary value problems (I) and (II) may be so numbered that the following arrangement of inequalities will hold good:

$$\ldots < \lambda_{-4}^{(2)} \leqslant \lambda_{-3}^{(2)} < \lambda_{-2}^{(1)} \leqslant \lambda_{-1}^{(1)} < \lambda_{-2}^{(2)} \leqslant \lambda_{-1}^{(2)} < \lambda_{-0}^{(1)} <$$
$$< \lambda_{+0}^{(1)} < \lambda_1^{(2)} \leqslant \lambda_2^{(2)} < \lambda_1^{(1)} \leqslant \lambda_2^{(1)} < \lambda_3^{(2)} \leqslant \lambda_4^{(2)} < \ldots$$

where

$$\lambda_{+0}^{(1)} = 0, \quad \lambda_{-0}^{(1)} < 0, \quad \text{if} \quad L > 0$$
$$\lambda_{-0}^{(1)} = 0, \quad \lambda_{+0}^{(1)} > 0, \quad \text{if} \quad L < 0$$

If $L = 0$, the law of alternation of these numbers will remain the same, provided the interval $(\lambda_{-0}^{(1)}, \lambda_{+0}^{(1)})$ is considered as being shrunk to the point zero, which is also a simple characteristic number of the boundary value problem (I).

All intervals having at their ends adjacent characteristic numbers of various boundary value problems (I) and (II) are zones of stability. The remaining intervals, including their ends, are zones of instability if they do not degenerate into a point. In the latter case the points will indicate stability, with the exception of the single point $\lambda = 0$.

In each unstable zone, with the exception of the interval $(\lambda_{-0}^{(1)},$

$\lambda_{+0}^{(1)}$), lies one and only one characteristic number of the boundary value problem

$$y''(t) + \lambda p(t) y(t) = 0, \qquad y(0) = y(\pi) = 0 \tag{0.4}$$

At present the basic part of this theorem is derived from a graph of function $\eta = A(\lambda)$ (see for instance [4]), as follows: any maximum of the curve $A(\lambda)$ is not lower than the line $\eta = 1$ and any minimum is not higher than the line $\eta = -1$, or, more precisely, if $\lambda = \lambda^*$, which is a stationary point of the function $A(\lambda)$, then

$$|A(\lambda^*)| \geqslant 1, \qquad A(\lambda^*) A''(\lambda^*) < 0 \tag{0.5}$$

According to Liapunov's theorem the parameter λ may assume arbitrarily large values (which will result in arbitrarily small values of frequency ω of parametric excitation), for which equation (0.2) will have unbounded solutions.

Consequently, the investigation of various problems concerning parametric resonance without allowing for resistance forces leads to paradoxical results. For instance, investigating the dunamic stability of bars by using linear equations without considering damping leads to the conclusion that for any small amplitude of a lingitudinally pulsating force there will be a small pulsating frequency at which the dynamic instability must take place. This approach to the problem of an attempt to determine zones of instability gives only a first approximation for the first zones of instability, so that in such a case it is impossible to determine the lower (or upper) limits of the values of unstable frequencies of the pulsating longitudinal forces.

This article defines certain values of the characteristic function and the characteristic indices of equation (0.2) for sufficiently large values of the parameter λ, at which values even a linear formulation of the problem indicates a scheme of computations free from the above paradoxes. As an example of how a paradox is encountered and how it may be avoided, we consider the dynamic stability of a prismatic rod with hinged supports.

Assuming for the sake of simplicity that function $p(t)$ in equation (0.2) is partially continuous, it should be noted that all results obtained here will hold good for any periodic function $p(t)$ which can be integrated, and may be specifically extended to cases of dynamic stability under periodically repeated longitudinal impact.

1. Investigation of the Increase of Characteristic Function $A(\lambda)$. Some ideas related to the theory of increase of complete functions [5,6] will be used subsequently.

From the work of Liapunov (see [7] and [8] p. 277) it follows that

$A(\lambda)$ is a complete function, where

$$|A(\lambda)| \leqslant \exp(\pi M^{1/2}|\lambda|^{1/2}), \qquad M = \max_{0 \leqslant t \leqslant \pi} |p(t \pm 0)| \tag{1.1}$$

The following proposition will now be proved.

Theorem 1. The index of convergence of the sequence of zeros of the characteristic function $A(\lambda)$ is equal to 1/2.

Proof. All zeros of function $A(\lambda)$ are real (see, for instance [4]). Let the sequence of zeros of function $A(\lambda)$ be designated by $\{a_n\}$ $(n = \pm 1, \pm 2, \pm 3, \ldots)$, arranged

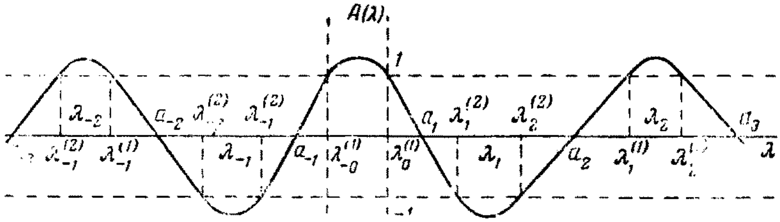

Fig. 1.

in order of increasing values of their modulus; let the positive values of a_n be associated with the positive indices and the negative value of a_n with the negative indices.

According to [4] and the Liapunov's theorem mentioned above, the following inequalities hold good:

$$\begin{aligned} \lambda_{n-1} < a_n < \lambda_n \qquad & \text{for } n = 1, 2, 3, \ldots \\ \lambda_n < a_n < \lambda_{n+1} \qquad & \text{for } n = -1, -2, -3, \ldots \end{aligned} \tag{1.2}$$

where λ_n is the characteristic number of the boundary value problem given by (0.4) (Fig. 1).

Let us first consider a case when $p(t) \geqslant 0$. In this case the asymptotic values apply (for instance see [9] p. 351)

$$\lim_{n \to \infty} \frac{n^2}{\lambda_n} = \frac{1}{\pi^2}\left(\int_0^\pi \sqrt{p(t)}\,dt\right)^2 \tag{1.3}$$

that is, when $n \to \infty$

$$\lambda_n = c^2 n^2 + o(n^2), \qquad c = \pi\left(\int_0^\pi \sqrt{p(t)}\,dt\right)^{-1} \tag{1.4}$$

The symbol $o(n^2)$ here means that $n^{-2}o(n^2) \to 0$ when $n \to \infty$.

According to equation (1.2), the following may also be obtained:

$$a_n = c^2 n^2 + o(n^2) \qquad (n \to \infty) \tag{1.5}$$

This leads to the conclusion that an infinite series having a general term $|a_n|^{-(1/2+\epsilon)}$ is convergent, but the one having a general terms $|a_n|^{-(1/2-\epsilon)}$ is divergent, which means that the sequence $\{a_n\}_1^\infty$ has the index of convergence $\beta = 1/2$.

Using $-\lambda$ instead of λ when $p(t) \leqslant 0$, and applying the same reasoning, the following asymptotic formula is obtained:

$$a_n = c_1^2 n^2 + o(n^2) \quad (n = -1, -2, -3, \ldots), \quad c_1 = \pi\left(\int_0^\pi \sqrt{-p(t)}\,dt\right)^{-1} \tag{1.6}$$

The theorem is thus proved for the case of a constant sign function.

Let it now be assumed that function $p(t)$ changes sign in the interval $(0, \pi)$. Let it have only one change of sign, for instance, $p(t) \geqslant 0$ when $(0 \leqslant t \leqslant t_0)$, and $p(t) \leqslant 0$ when $(t_0 \leqslant t \leqslant \pi)$.

If another boundary condition $y(t_0) = 0$ is added to the boundary value problem expressed by equation (0.4), then, according to the theorem of frequency changes when the above additional boundary condition is applied (for instance see [10] Ch. 5), $\lambda_n \leqslant \lambda_n^* \leqslant \lambda_{n+1}$ when $n = 1, 2, 3 \ldots$, and $\lambda_{n-1} \leqslant \lambda_n^* \leqslant \lambda_n$ when $n = -1, -2, -3 \ldots$, where λ_n^* is the characteristic number of the boundary value problem (0.4) with boundary condition $y(t_0)=0$. Then, according to equation (1.2), $\lambda_{n-2}^* < a_n < \lambda_n$ when $n = 3, 4, 5 \ldots$ and $\lambda_n^* < a_n < \lambda_{n+2}$ when $n = -3, -4, -5 \ldots$ However, the positive and negative characteristic numbers are respectively the characteristic numbers of the following two boundary value problems:

$$-y''(t) = \lambda p(t) y(t), \qquad y(0) = y(t_0) = 0$$

$$-y''(t) = \lambda p(t) y(t), \qquad y(t_0) = y(\pi) = 0$$

for which the asymptotic values (1.5) and (1.6) hold good:

$$a_n = c_+^2 n^2 + o(n^2) \qquad (n = 1, 2, 3, \ldots)$$

$$-a_n = c_-^2 n^2 + o(n^2) \qquad (n = -1, -2, -3, \ldots) \tag{1.7}$$

where

$$c_+ = \pi\left(\int_0^{t_0} \sqrt{p(t)}\,dt\right)^{-1} = \pi\left(\int_0^\pi \sqrt{p_+(t)}\,dt\right)^{-1}$$

$$c = \pi\left(\int_{t_0}^\pi \sqrt{p(t)}\,dt\right)^{-1} = \pi\left(\int_0^\pi \sqrt{p_-(t)}\,dt\right)^{-1}$$

In these equations

$$p_+(t) = {}^1/_2\{|p(t)| + p(t)\}, \qquad p_-(t) = {}^1/_2\{|p(t)| - p(t)\}$$

Following similar reasoning, it is also possible to obtain asymptotic formulas for the zeros of function $A(\lambda)$ when function $p(t)$ changes sign any finite number of times in the interval $(0, \pi)$. Thus in this case the sequence of zeros $\{a_n\}_1^\infty$ of the function $A(\lambda)$ also has the index of convergence $\beta = 1/2$.

Theorem 2. The order of increase of function $A(\lambda)$ is equal to 1/2.

Proof. From (1.1) it follows that the order of increase of function $A(\lambda)$ is not larger than 1/2. On the other hand, since the index of con- convergence of the sequence of zeros of the complete function does not exceed the order of increase of such a function, then, allowing for Theorem 1, the order of increase of function $A(\lambda)$ is not less than 1/2. This proves the theorem.

It follows from Theorem 2 that $A(\lambda)$ may be expressed as

$$A(\lambda) = \prod_{n=-\infty}^{\infty}\left(1 - \frac{\lambda}{a_n}\right) \tag{1.8}$$

Here the known relation $A(0) = 1$ was used.

2. The Asymptotic Values of the Characteristic Function and of the Real Part of Characteristic Exponents. Let us consider two complete functions

$$P(\lambda) = \prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{a_n}\right), \qquad Q(\lambda) = \prod_{n=1}^{\infty}\left(1 + \frac{\lambda}{b_n}\right) \tag{2.1}$$

where $a_n (n = 1, 2, 3 \dots)$ are positive zeros and $b_n = -a_n (n = -1, -2, -3 \dots)$ are absolute values of the negative zeros of the function $A(\lambda)$.

According to this, the infinite products (2.1) are convergent and

$$A(\lambda) = P(\lambda)Q(\lambda) \tag{2.2}$$

Let the number of zeros of the function $P(\lambda)$, in a circle of radius r, be designated by $N_p(r)$. Also let $r > a_1$; then, for a certain N, we have

$$a_N \leqslant r < a_{N+1} \quad \text{or} \quad c_+^2 N^2(r) + o(N^2) \leqslant r < c_+^2 [N(r) + 1]^2 + o([N + 1]^2)$$

therefore

$$\lim_{r\to\infty} \frac{N_p(r)}{r^{1/2}} = \frac{1}{c_+} = \frac{1}{\pi}\int_0^\pi \sqrt{p_+(t)}\,dt \tag{2.3}$$

Also, designating the number of zeros of the function $Q(\lambda)$ in a circle of radius r by $N_Q(r)$, we may similarly obtain

$$\lim_{r\to\infty} \frac{N_Q(r)}{r^{1/2}} = \frac{1}{c_-} = \frac{1}{\pi}\int_2^{\pi} \sqrt{p_-(t)}\,dt \tag{2.4}$$

If for the sequence $r_1, r_2, r_3 \ldots$ the relation $N(r)\ \ cr^{\beta}$ holds good, $N(r)$ being the number of terms of the sequence in the circle of radius r and β the exponent of convergence of the sequence, then such a sequence is called regular. Since $\lim r^{-\beta}N_p(r)$ and $\lim r^{-\beta}N_Q(r)$ exist when $r \to \infty$ ($\beta = 1/2$), then the sequences $\{a_n\}_1^{\infty}$ and $\{b_n\}_1^{\infty}$ are regular.

Thus, for the functions $Q(\lambda)$ and $P(\lambda)$ the following expressions for the limits will apply (see [11] p. 20):

$$\lim_{r\to\infty} \frac{\ln Q(re^{i\vartheta})}{N_Q(r)} = \pi \exp\frac{i\vartheta}{2} \qquad (-\pi < \vartheta < \pi)$$

$$\lim_{r\to\infty} \frac{\ln P(re^{i\vartheta})}{N_p(r)} = \pi \exp\frac{i(\vartheta-\pi)}{2} \qquad (0 < \vartheta < 2\pi)$$

Utilizing relations (2.3) and (2.4), it is established that

$$\lim_{r\to\infty} \frac{\ln Q(re^{i\vartheta})}{r^{1/2}} = \exp\left(\frac{i\vartheta}{2}\right)\int_0^{\pi} \sqrt{p_-(t)}\,dt$$

$$\lim_{r\to\infty} \frac{\ln P(re^{i\vartheta})}{r^{1/2}} = \exp\left(\frac{i(\vartheta-\pi)}{2}\right)\int_0^{\pi} \sqrt{p_+(t)}\,dt$$

where the first limit is for $(-\pi < \theta < \pi)$ and the second for $(0 < \theta < 2\pi)$. Therefore, the index of increase of function $A(\lambda)$ along the ray $\arg\lambda = \theta$

$$h_Q(\vartheta) = \overline{\lim_{r\to\infty}} \frac{\ln|Q(re^{i\vartheta})|}{r^{1/2}} = \cos\frac{\vartheta}{2}\int_0^{\pi} \sqrt{p_-(t)}\,dt \qquad (-\pi < \vartheta < \pi)$$

and since the index of increase is the periodic continuous function θ of the period 2π; the relation obtained holds good for any θ if $|\cos 1/2\ \theta|$ is substituted for $\cos 1/2\ \theta$.

Analogously, one finds that the index of increase of function $P(\lambda)$ along the ray $\arg\lambda = \theta$ is

$$h_P(\vartheta) = \left|\sin\frac{\vartheta}{2}\right|\int_0^{\pi} \sqrt{p_+(t)}\,dt$$

Since the increase of functions $P(\lambda)$ and $Q(\lambda)$ is regular, that is, there exist

$$\lim r^{-1/2}\ln|P(re^{i\vartheta})| \text{ and } \lim r^{-1/2}\ln|Q(re^{i\vartheta})| \qquad \text{for } r\to\infty$$

then, according to (2.2), for the index of increase $h_A(\theta)$, of the function $A(\lambda)$, we obtain

$$h_A(\vartheta) = h_Q(\vartheta) + h_P(\vartheta) = \left|\cos\frac{\vartheta}{2}\right| \int_0^{\pi} \sqrt{p_-(t)}\,dt + \left|\sin\frac{\vartheta}{2}\right| \int_0^{\pi} \sqrt{p_+(t)}\,dt \quad (2.5)$$

and consequently, along the positive direction of the real axis,

$$h_A(0) = \int_0^{\pi} \sqrt{p_-(t)}\,dt$$

The value obtained for $h_A(0)$ permits the following proposition to be stated.

Theorem 3. For any $\epsilon > 0$ it is possible to find such $R(\epsilon) > 0$, that for all λ, satisfying the condition $\lambda > R(\epsilon)$, the following estimate holds:

$$|A(\lambda)| < \exp\left\{\left(\int_0^{\pi} \sqrt{p_-(t)}\,dt + \varepsilon\right)\sqrt{\lambda}\right\} \quad (2.6)$$

In the inequality (2.6) the value of the integral may not be replaced by any smaller value.

From this proposition follows the corollary below.

Corollary. For any $\epsilon > 0$ it is possible to find $R(\epsilon)$ such that for all $\lambda > R(\epsilon)$ the following estimate holds good:

$$-\left(\frac{1}{\pi}\int_0^{\pi} \sqrt{p_-(t)}\,dt + \varepsilon\right)\sqrt{\lambda} < \alpha(\lambda) < \left(\frac{1}{\pi}\int_0^{\pi} \sqrt{p_-(t)}\,dt + \varepsilon\right)\sqrt{\lambda} \quad (2.7)$$

where $\alpha(\lambda)$ is the real part of the characteristic exponent.

$$\varkappa(\lambda) = \alpha(\lambda) + i\beta(\lambda)$$

3. Equation with Damping. Let us consider equation

$$y''(t) + 2\mu\nu y'(t) + \mu^2 q(t)\,y(t) = 0 \quad (3.1)$$

where $\nu > 0$ is a constant coefficient of damping, $\mu > 0$ is a real coefficient, $q(t) \geqslant 0$ is a real function of t having period π and continuous in the segment $(0, \pi)$. This equation differs from (1.2) by the introduction of the damping factor $\mu\nu$, which increases with μ. Such an equation may be called a "damped" equation.

The solutions of (3.1), bounded on the real semi-axis, will be called stable. Bounded solutions based on the condition that $\lim y(t) = 0$ when $t \to \infty$ will be called asymptotically stable.

If $\nu^2 \geqslant \max q(t)$ $(0 \leqslant t \leqslant \pi)$, then it follows from [12] that the solutions of (3.1) are stable for any value of μ; and when $\min q(t) > 0$ $(0 \leqslant t \leqslant \pi)$, the solutions are asymptotically stable. If $\nu^2 < \max q(t)$ $(0 \leqslant t \leqslant \pi)$, the solutions may be unstable (for certain values of parameter λ). However, for such a case the following theorem may be formulated.

Theorem 4. If the function $q(t) \geqslant 0$ is not identically equal to zero in the interval $(0, \pi)$, then for any $\nu > 0$ there will be $R(\nu) > 0$ such that for $\mu > R(\nu)$ all solutions of (3.1) are asymptotically stable.

Proof. By substitution $y(t) = u(t)\exp(-\mu\nu t)$ the equation may be reduced to the following form:

$$u''(t) + \mu^2[q(t) - \nu^2]u(t) = 0 \tag{3.2}$$

Obviously, the only case which need be considered is when μ^2 is inside the unstable zone of (3.2), i.e., when $A^2(\mu^2) > 1$. In such a case any solution of (3.1) may be given in the following form:

$$y(t) = e^{-\mu\nu t}[c_1 e^{\alpha(\mu^2)t} f_1(t) + c_2 e^{-\alpha(\mu^2)t} f_i(t)]$$

where c_1 and c_2 are constants determined by the initial conditions, $f_1(t)$ and $f_2(t)$ are certain periodic functions and $\alpha(\mu^2)$ is a real part of the characteristic exponent which lies in the right half-plane. Thus the theorem will be proved if it can be shown that for sufficiently large values of μ the inequality $\alpha(\mu^2) < \mu\nu$ holds good.

It follows from the corollary of Theorem 3 that for any $\epsilon > 0$ there is $R(\epsilon) > 0$ such that for all $\mu > R(\epsilon)$ there will be

$$\alpha(\mu^2) < \left(\frac{1}{\pi}\int_0^\pi \sqrt{p_-(t)}\,dt + \varepsilon\right)\mu, \qquad p_-(t) = \begin{cases} \nu^2 - q(t) & \text{for } \nu^2 \geqslant q(t) \\ 0 & \text{for } \nu^2 \leqslant q(t) \end{cases} \tag{3.3}$$

In conformity with condition $q(t) > 0$,

$$\nu^2 \geqslant \max p_-(t) \qquad (0 \leqslant t \leqslant \pi) \tag{3.4}$$

where the equal sign holds good only for those values of t for which $q(t) = 0$.

On the other hand

$$\max\sqrt{p_-(t)} \geqslant \frac{1}{\pi}\int_0^\pi \sqrt{p_-(t)}\,dt \qquad (0 \leqslant t \leqslant \pi) \tag{3.5}$$

where the equal sign holds good only when $p_-(t) \equiv \text{const}$ (consequently, $q(t) \equiv \text{const}$) in the interval $(0, \pi)$.

Comparing (3.4) and (3.5), we may conclude that if $q(t) \not\equiv 0$ in the interval $(0, \pi)$, then

$$\nu^2 - \frac{1}{\pi}\int_0^\pi \sqrt{p_-(t)}\,dt > 0$$

This difference will be designated by $\epsilon(\nu)$. According to (3.3) there will exist $R(\epsilon(\nu)) = R(\nu)$ such that

$$\alpha(\mu^2) < \left\{\frac{1}{\pi}\int_0^\pi \sqrt{p_-(t)}\,dt + \varepsilon(\nu)\right\}\mu = \mu\nu \qquad \text{for } \mu > R(\nu)$$

This proves the theorem*.

Corollary. Equation (3.1) either has no zones of instability (as for instance when $\nu^2 \geqslant \max q(t)$ $(0 \leqslant t \leqslant \pi)$, or it has only a finite number of unstable zones.

4. On a Possible Paradox in the Problems of Dynamic Stability. Let us consider the usual problem of dynamic stability of a pin-supported prismatic rod under the action of longitudinal periodic forces $p(t)$ (Fig. 2), first investigated by Beliaev [14]. This problem leads to the investigation of the stability of solutions of the differential equation

$$EI\frac{\partial^4 y}{\partial x^4} + P(t)\frac{\partial^2 y}{\partial x^2} + \frac{\gamma F}{g}\frac{\partial^2 y}{\partial t^2} = 0 \tag{4.1}$$

with boundary conditions

$$y(0, t) = y(l, t) = 0, \qquad \frac{\partial^2 y(0, t)}{\partial x^2} = \frac{\partial^2 y(l, t)}{\partial x^2} = 0$$

Here EI is the stiffness, γ is the specific weight, F is the cross-sectional area, g is the acceleration of gravity. Following Beliaev, let us consider a special form of the function $P(t) = P_0 \cos \omega t$. We find the

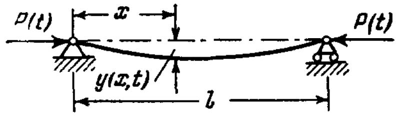

Fig. 2.

solution in the form of a series

$$y(x, t) = T_1(t)\varphi_1(x) + T_2(t)\varphi_2(x) + \dots$$

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* It should be noted that when $q(t)$ is not reduced to zero and has a bounded derivative, the asymptotic stability at large values of μ follows from the work Leonov [13]. In this case

where $\{\phi_n(x)\}$ is the normalized system of the fundamental functions characterizing the problem of natural vibrations of the same rod; these vibrations coincide with the fundamental functions dealing with the problem of static stability; then, considering the independent variable $\tau = 1/2\, \omega t$, we find the following equations for the Fourier coefficients $T_n(\tau)$:

$$T_n''(\tau) + \frac{4k_n^2}{\omega^2}(1 - b_n \cos 2\tau) T_n(\tau) = 0, \qquad k_n = \frac{n^2\pi^2}{l^2}\sqrt{\frac{gEI}{\gamma F}}$$

$$b_n = P_0 / P_n, \qquad P_n = EIn^2\pi^2 l^{-2} \qquad (n = 1, 2, 3, \ldots) \tag{4.2}$$

where k_n is the natural frequency of vibrations and P_n is Euler's critical forces of the given rod.

Equation (4.2) is of the form of (0.2), in which $\lambda = 4k_n^2\omega^{-2}$ and $p(t) = 1 - b_n \cos 2t$. According to Liapunov's theorem, even when $P_0 < P_n$ there is an infinite number of unstable zones, which develop with the infinitely increasing values of parameter λ. Next, considering parameter ω, we come to the following conclusion:

There exist an infinite number of series (for $n = 1, 2, 3 \ldots$) for unstable zones for the frequency of the periodic longitudinal force equal to ω; also in each series there is an infinite number of zones of instability which concentrate to the point $\omega = 0$.

The determination of the boundaries of such zones of instability for various equations (when $n = 1, 2, 3 \ldots$) leads to the diagram shown on Fig. 3, where

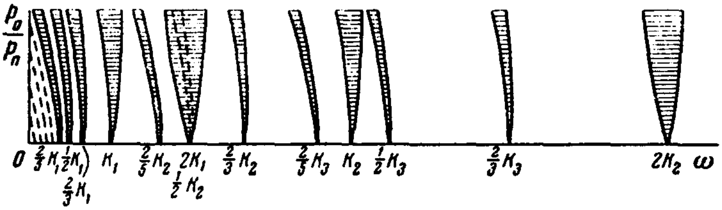

Fig. 3.

the zones of unstable frequencies for the first three equations (4.2) are indicated. This distribution of unstable zones leads to the paradoxical conclusions mentioned above.

Such a paradox is the result of extreme idealization in the formulation of dynamic stability problems.

However, even in the frame work of the linear treatment of the problem, the paradox may be removed by allowing for damping forces. It may be assumed (for instance see [15]), that there are external resistant forces proportional to the speed of rod particles, and internal resistant forces proportional to the speed of deformation. With these assumptions equations (4.2) take on the following form:

$$T_n''(\tau) + \frac{2k_n}{\omega}\left(\frac{\xi k_n}{E} + \frac{\eta}{k_n}\right) T_n'(\tau) + \frac{4k_n^2}{\omega^2}(1 - b_n \cos 2\tau)\, T_n(\tau) = 0 \qquad (4.3)$$

$$(n = 1,\ 2,\ 3, \ldots$$

where ξ is the coefficient of internal friction, and η is the coefficient of external friction; the meaning of other symbols remains the same. This leads to equations (3.1) with damping, where

$$\mu_n = \frac{2k_n}{\omega}, \qquad \nu_n = \frac{1}{2}\left(\frac{\xi k_n}{E} + \frac{\eta}{k_n}\right), \qquad q_n(t) = 1 - b_n \cos 2t$$

If only external resistance exists (i.e. when $\xi = 0$), for each equation of type (3.1) of the infinite system (4.3) there will be a series of a finite number of unstable zones, as follows from Theorem 4. However, since there is an infinite number of such series and a rapid reduction of damping coefficient ν_n with the increase of the index n, allowing for the external resistance alone does not resolve the said paradox*.

If $\nu_n^2 > \max q_n(t) = 1 + b_n$, which takes place when

$$n^2 \geqslant \frac{l^2 \sqrt{\gamma EF}}{\pi^2 \xi \sqrt{gI}}\left(1 + \sqrt{1 + \frac{2P_0 \xi \sqrt{g}}{E\sqrt{\gamma FEI}}}\right)$$

then, according to Theorem 4, the unstable zones will be absent, i.e. with the inclusion of linear internal damping there will be only a finite number of series of instability and only a finite number of unstable zones in each series. Thus when $q(t) \geqslant 0$, and the frequency of the longitudinal force is low enough, the rod is always stable. Since exact estimates have been considered here, it may be said that if $q_n(t)$ changes sign, then instability will occur for sufficiently low frequencies. The function $q_n(t) = 1 - P(t)/P_n$ may change sign only when there are values of $P(t)$ larger than the critical force; thus if $\max P(t) > P_1$, then with sufficiently slow changes in the magnitude of applied force, instability of the rod occurs; this accords with experimental results.

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* Beilin and Dzhanelidze, in their review of works on the dynamic stability of rigid systems, doubt whether the paradox can be resolved by considering only external linear resistance. They do not allow for the fact that there is an infinite number of series having unstable zones when only external resistance is present.

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